Advanced Learners and the new NC SCOS

Advanced Math and Science Content for Teachers of Advanced Learners

MATH SESSION # 3

Using Recursion to Explore Real World Problems Part II

Grades 7 - 12

Date: December 4, 2012

Developed in partnership with DPI~AIG and NC School of Science and Mathematics

Advanced Content for Teachers of Advanced Learners

- Why?
  - To ensure the growth of advanced learners
  - To develop teachers' understanding of advanced math/science content and instructional practices

- What?
  - 14 Content-based PD sessions, webinar and archived; 7 mathematics, 7 science

Using Recursion to Explore Real World Problems Part II

Goals: To extend the topic of recursion to model more complex real-world scenarios and to use more advanced math concepts to solve these extended problems. The webinar's content extends the Common Core State Standards in Math that are listed on the next slide.

Recursion & NC Standard Course of Study

- F-BF.1. Write a function that describes a relationship between two quantities.
  - Determine an explicit expression, a recursive process, or steps for calculation from a context.
- F-BF.2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.
- F-IF.4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

As we work through the problems:

- Consider how your student will approach the problem.
- Anticipate their responses and questions.
- Consider how we can use their ideas and contributions to build the mathematical concepts.

From last time

Example 3: Fish and Wildlife

The Fish and Wildlife Division monitors the trout population in a stream that is under its jurisdiction. Its research indicates that natural predators, together with pollution and fishing, are causing the trout population to decrease at a rate of 20% per month. The Division proposes to introduce additional trout each month to replenish the stream. Assume the current population is 300. Use recursive equations, tables and graphs to investigate the following questions.

Example 3: Fish and Wildlife

1. What will happen to the trout population over the next 10 months with no replenishment program?

2. What is the long-term result of introducing 100 trout into the stream each month?

Ties to the Candy Problem and the idea of equilibrium.

Recursive Equations for the Fish Problem

$$F(0) = 300$$

$$F(1) = F(0) - 0.2F(0) + 100$$

$$F(2) = F(1) - 0.2F(1) + 100$$

$$\vdots$$

$$F(n) = F(n-1) - 0.2F(n-1) + 100$$

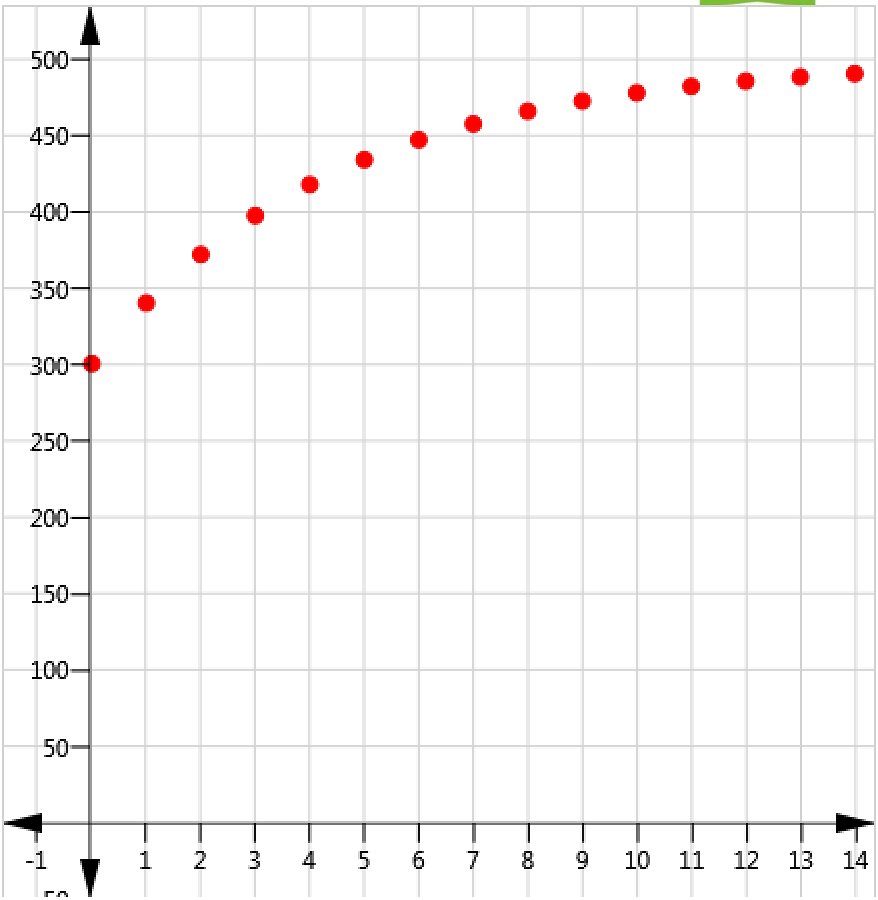
or

$$F(n) = 0.8F(n-1) + 100$$

Graphing the Fish Over Time

$$F(n) = 0.8F(n-1) + 100$$

Notice that the graph appears to level off or reach an **equilibrium** value.



Can we predict the equilibrium level without iterating?

Consider the recursive equations. When the amount added is the same as the amount lost, we reach equilibrium (like the candy problem).

$$F(n) = F(n-1) - 0.2F(n-1) + 100$$

$$F(n) = F(n-1) - 0.2F(n-1) + 100$$

lost added

or

$$F(n) = 0.8F(n-1) + 100$$

Solving for Equilibrium Analytically

Let's set these equal and solve for F(n-1)

$$F(n) = F(n-1) - 0.2F(n-1) + 100$$

lost

added

$$0.2F(n-1) = 100$$

$$F(n-1) = \frac{100}{0.2} = 100 \times 5 = 500$$

so **500** fish is the equilibrium level.

Solving for Equilibrium Analytically

Or we can think about the fact that when we reach equilibrium, F(n) = F(n-1). So we have…

$$F(n) = 0.8F(n) + 100$$

$$0.2F(n) = 100$$

$$F(n) = \frac{100}{0.2} = 500$$

Some Interesting Observations:

- We can predict the equilibrium without iterating.
- The equilibrium value does not depend on the initial population.
- If we want to reach a certain equilibrium value, we can figure out how many fish to add each month.

Another good example of this is when a doctor prescribes a dose of a drug, they are trying to achieve a therapeutic level of the drug in your system.

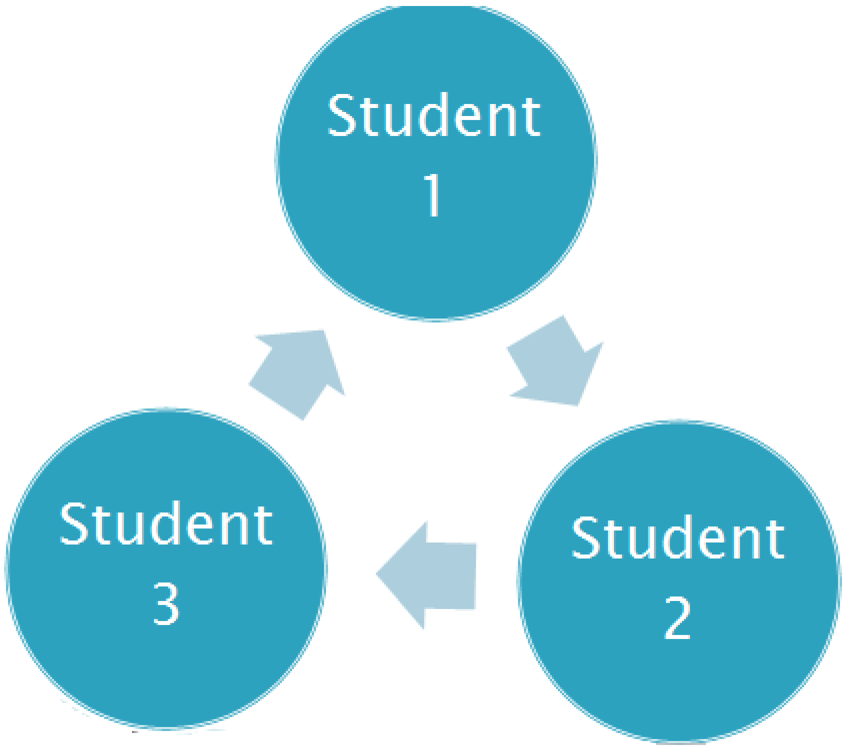
Revisit The Candy Problem

Pass the Candy

- We will form a group of 3 or 4 people. Appoint a record keeper.
- Each member will get a bag with some candy. **Count** the number of pieces of candy in your bag and tell the record keeper that number so they can record it on the Tally Sheet.
- When I say "Pass the Candy", we will each pass half of our candy to the person on our left. We will count the number of pieces of candy and record that number on the Tally Sheet.

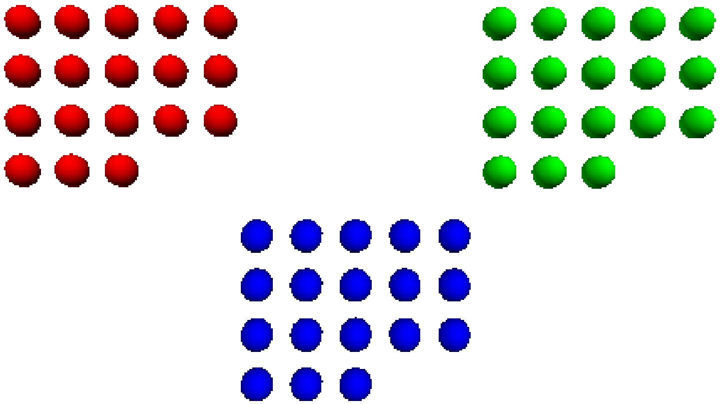
Student 1 passes to Student 2, Student 2 passes to student 3 and Student 3 Passes to Student 1

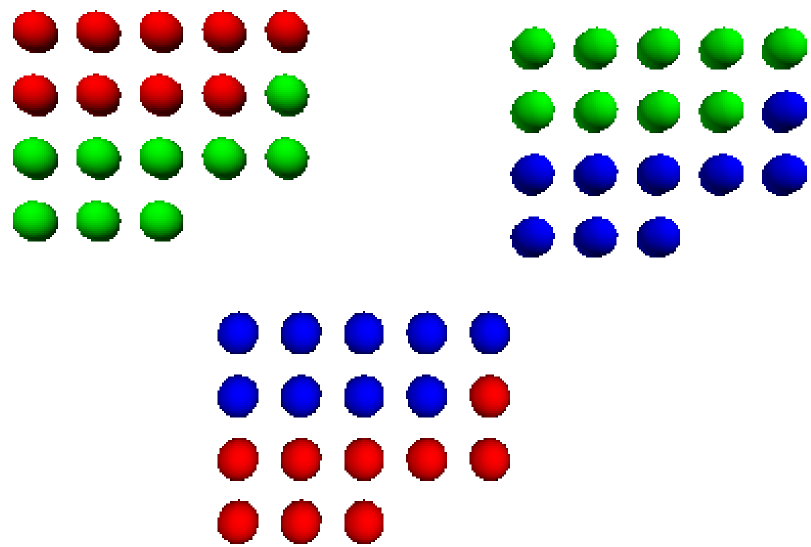
Recall we reached equilibrium when the number of candies we were passing was the same as the amount we were receiving.



See the Python simulation on next slide.

Recall we reached equilibrium when the number of candies we were passing was the same as the amount we were receiving.





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[R  G  B ]
[22  2 30]
[26 12 16]
[21 19 14]
[18 20 16]
[17 19 18]
[18 18 18]
[18 18 18]
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Loan Problem

You want to buy a car and take out a loan in the amount of $5,000 at 0.75% monthly interest. How long would it take to pay off the loan if you paid back $175 each month?

Let's write the recursive equations for this scenario. Let B be the balance of the loan each month.

$$B(0) = 5000$$

$$B(1) = B(0) + .0075B(0) - 175$$

$$B(2) = B(1) + 0.0075B(1) - 175$$

$$\vdots$$

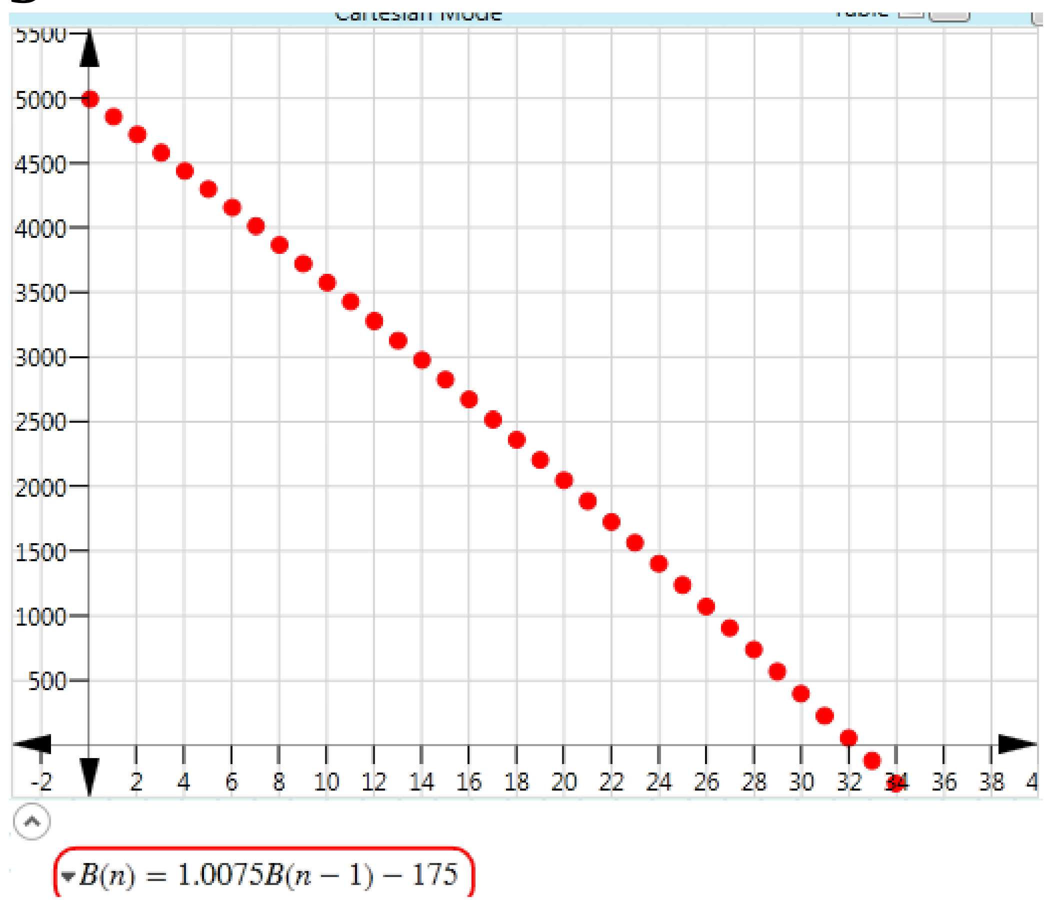
$$B(n) = B(n-1) + 0.0075B(n-1) - 175$$

Use the calculator to generate balance at the end of each month.

Graph of Balance Over Time

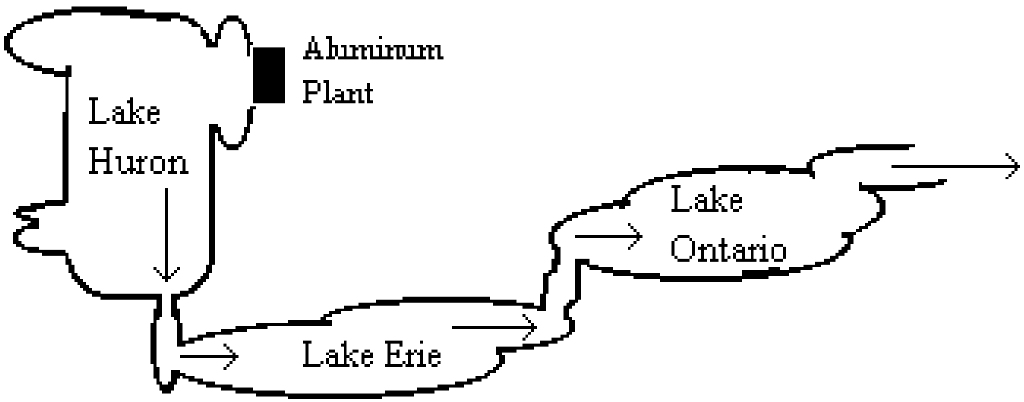
$$B(n) = 1.0075B(n-1) - 175$$

Takes about 32 months to pay off the loan.



Great Lakes Problem

Most of the water flowing into Lake Erie comes from Lake Huron, and most of the water flowing into Lake Ontario is from Lake Erie. Each year, 11% of the water in Lake Huron flows into Lake Erie, while 36% of the water in Lake Erie flows into Lake Ontario, and 12% of the water in Lake Ontario flows out to the sea.



Source: Intermath: Four Sample Problems, COMAP, Inc., Lexington, MA, 1992.

Great Lakes Problem

For generations, factories on the lakes had been dumping a pollutant into the water. Presently, there are 4000 units of pollutant in Lake Huron, 2000 units in Lake Erie, and 3000 units in Lake Ontario. For the most part, this form of pollution has stopped. Only two such factories remain. One, on Lake Huron, is dumping 25 units of pollutant into the water each year; the other on Lake Ontario is dumping 20 units of the pollutant into the water each year.

- Will the amount of pollutant in each of the three lakes ever be reduced to 10% of its present level? If so, when.
- What is the long-term level of pollutant in the lakes?

Writing the Recursive Equations

$$H_0 = 4000 \qquad E_0 = 2000 \qquad O_0 = 3000$$

$$H_n = H_{n-1} - 0.11H_{n-1} + 25$$

$$E_n = E_{n-1} - 0.36E_{n-1} + 0.11H_{n-1}$$

$$O_n = O_{n-1} - 0.12O_{n-1} + 0.36E_{n-1} + 20$$

$$H_n = 0.89H_{n-1} + 25$$

$$E_n = 0.64E_{n-1} + 0.11H_{n-1}$$

$$O_n = 0.88O_{n-1} + 0.36E_{n-1} + 20$$

Great Lakes Problem

- Nice project for advanced students. They may even pose some of their own questions or do some research on current news articles about pollution levels in the Great Lakes.

- You may want to use other technological tools like spreadsheets to explore these more complex problems.

- Can extend this problem to a AP BC Calculus Lab. We can write coupled differential equations and then solve the DEs using Euler's Method.

Develop Closed Form for Mixed Recursion

- Can be challenging.
- Your students need to be familiar with the formula for the sum of a geometric series.
- Can use the loan payment problem as motivation – if you are looking for the monthly payment to pay off an *n*-year loan.

The slides in the Appendix contain the mathematical derivation for the closed form for the Fish Problem.

Using our students ideas to create the mathematics…

Reflect on the ideas we have discussed and consider how we have asked students to be part of the process in developing the mathematical concepts.

Thank You!

Join Us for the Next Webinar

READY

Session 4: Introduction to Parametric Equations

We can use linear and quadratic equations to model the motion of various objects in the plane. In this session we will use parametric equations to build a model for the motion of a Mars rover. Then we can use the distance formula to create a model to find out how close the rover gets to its intended target. We will use the TI calculator to graph the parametric equations and the distance function.

Tuesday, January 22, 2013

3:45 – 4:45

Resources

- Links for NCSSM Recursion Materials

http://www.dlt.ncssm.edu/stem/content/lesson-1-introduction-recursion
http://www.dlt.ncssm.edu/stem/lesson-2recursion

- Links for DPI Math Resources (HS Math Wiki)

http://maccss.ncdpi.wikispaces.net/High+School

- Sneha Shah-Coltrane

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Appendix A

The next few slides contain the derivation for the closed form for mixed recursion for the Fish Problem.

We write an explicit function for this type of recursive equation.

$$F(0) = 300$$

$$F(1) = 0.8F(0) + 100$$

$$F(2) = 0.8F(1) + 100 = 0.8\left[0.8F(0) + 100\right] + 100$$

$$= 0.8^2 F(0) + 0.8 \times 100 + 100$$

$$F(3) = 0.8F(2) + 100 = 0.8\left[0.8^2 F(0) + 0.8 \times 100 + 100\right] + 100$$

$$= 0.8^3 F(0) + 0.8^2 \times 100 + 100 \times 0.8 + 100$$

$$= 0.8^3 F(0) + 100\left[0.8^2 + 0.8 + 1\right]$$

Derivation of Closed Form Continued

$$F(4) = 0.8^4 F(0) + 0.8^3 \times 100 + 100 \times 0.8^2 + 100 \times 0.8 + 100$$

$$= 0.8^4 F(0) + 100\left[0.8^3 + 0.8^2 + 0.8 + 1\right]$$

Geometric Series

$$F(n) = 0.8^n F(0) + 0.8^{n-1} \times 100 + \ldots + 100 \times 0.8^2 + 100 \times 0.8 + 100$$

Geometric Series

Almost there…

$$F(n) = 0.8^n F(0) + 0.8^{n-1} \times 100 + \ldots + 100 \times 0.8^2 + 100 \times 0.8 + 100$$

Geometric Series

The formula for the sum of a finite geometric series is

$$S = 1 + R + R^2 + R^3 + \ldots + R^N = \frac{1 - R^{N+1}}{1 - R}$$

Almost there…

$$F(n) = 0.8^n F(0) + 0.8^{n-1} \times 100 + \ldots + 100 \times 0.8^2 + 100 \times 0.8 + 100$$

Geometric Series

$$S = 1 + R + R^2 + R^3 + \ldots + R^N = \frac{1 - R^{N+1}}{1 - R}$$

$$F(n) = 0.8^n F(0) + 0.8^{n-1} \times 100 + \ldots + 100 \times 0.8^2 + 100 \times 0.8 + 100$$

$$= 0.8^n F(0) + 100 \left[\frac{1 - 0.8^{n+1}}{1 - 0.8} \right]$$